[redacted]

N64 13276
CODE-1
CR-53007

Technical Report No. 32-479

Thermoelastic Analysis of a Parabolic Shell

G. S. Stern

JPL
JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

August 1, 1963

~~Technical Report~~ No. 32-479

Thermoelastic Analysis of a Parabolic Shell

G. S. Stern 1 Aug. 1963 17p refs

2

(NASA Contract NAS7-100)

X OTS (NASA CR-53007; JPL-TR-32-479) OTS: $1.60ph $0.80mf

R. R. McDonald

R. R. McDonald, Chief
Engineering Research

JET PROPULSION LAB~~ORATORY~~,
CALIF~~ORNIA~~, INST~~ITUTE~~ OF TECH~~NOLOGY~~,
PASADENA~~, CALIFORNIA~~

~~August 1, 1963~~

Prepared Under Contract No. NAS 7-100
National Aeronautics & Space Administration

CONTENTS

FIGURES

ABSTRACT

13276

The differential equations of equilibrium of a thin, homogeneous, isotropic, elastic shell of revolution subjected to axisymmetric thermal loading are established. In particular, the results are specialized to the case of a parabolic shell. An approximate solution of these equations is found by the method of asymptotic integration. As an illustrative example, the stresses and rotation are computed for the case of a parabolic shell with an attached edge ring, subjected to a thermal gradient through the thickness of the shell.

AUTHOR

I. GENERAL THEORY OF STEADY-STATE STRESS IN A THIN SHELL OF REVOLUTION SUBJECTED TO AXISYMMETRIC THERMAL LOADING

The Duhamel-Neumann law (Ref. 1) states that the strains in an elastic body undergoing thermal expansion are related to the stresses by

$$\epsilon_\phi = \frac{1}{E}(\sigma_\phi - \nu\sigma_\theta) + \alpha T \tag{1}$$

$$\epsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_\phi) + \alpha T \tag{2}$$

$$\epsilon_{\phi z} = \frac{1+\nu}{E}\sigma_{\phi z} \tag{3}$$

where T represents the difference between the elevated and equilibrium temperatures in the shell, and will be assumed to be an arbitrary function of the polar angle ϕ, and a linear function of the thickness variable z (see Fig. 1). It will be assumed throughout this Report that the variations in temperature are sufficiently small so that the corresponding variations in the elastic moduli and coefficient of thermal expansion are negligible.

The strains at a point in the shell are given in terms of midplane strains by

$$\epsilon_\phi = \frac{1}{1+\dfrac{z}{r_1}}(\bar{\epsilon}_\phi + z\kappa_\phi) \tag{4}$$

$$\epsilon_\theta = \frac{1}{1+\dfrac{z}{r_2}}(\bar{\epsilon}_\theta + z\kappa_\theta) \tag{5}$$

where ϵ_ϕ, ϵ_θ are the midplane strains.

The midplane strains are expressed in terms of displacements (see Fig. 1) as follows:

$$\bar{\epsilon}_\phi = \frac{1}{r_1}\left(\frac{dv}{d\phi} - w\right) \tag{6}$$

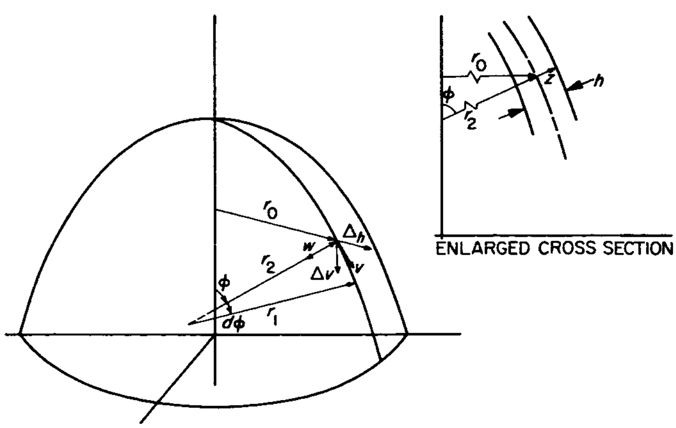

Fig. 1. Coordinate and displacement sign convention

$$\bar{\epsilon}_\theta = \frac{1}{r_2}(v \cot \phi - w) \tag{7}$$

$$\kappa_\phi = \frac{1}{r_1}\frac{dV}{d\phi} \tag{8}$$

$$\kappa_\theta = \frac{1}{r_2} V \cot \phi \tag{9}$$

where V represents the angle of rotation of a tangent to a meridian and is given in terms of displacement by

$$V = \frac{1}{r_1}\left(v + \frac{dw}{d\phi}\right) \tag{10}$$

Solving Eq. (1) and (2) for stresses yields

$$\sigma_\phi = \frac{E}{1-\nu^2}(\epsilon_\phi + \nu\epsilon_\theta) - \frac{E\alpha T(\phi,z)}{1-\nu} \tag{11}$$

$$\sigma_\theta = \frac{E}{1-\nu^2}(\epsilon_\theta + \nu\epsilon_\phi) - \frac{E\alpha T(\phi,z)}{1-\nu} \tag{12}$$

The stress resultants (see Fig. 2) are defined by

$$N_\phi = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_\phi \left(1 + \frac{z}{r_2}\right) dz \tag{13}$$

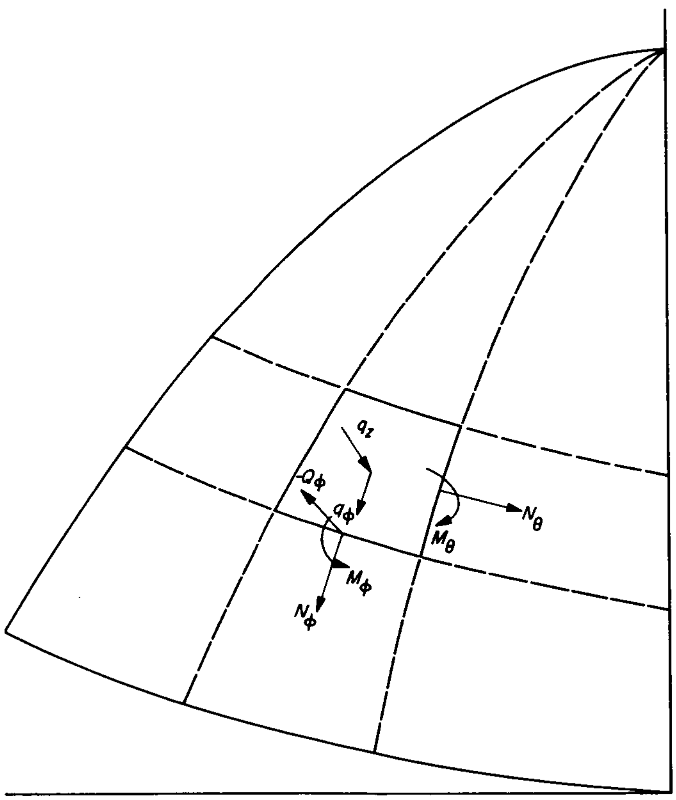

Fig. 2. Force sign convention

$$N_\theta = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_\theta \left(1 + \frac{z}{r_1}\right) dz \tag{14}$$

$$M_\phi = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_\phi \left(1 + \frac{z}{r_2}\right) dz \tag{15}$$

$$M_\theta = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_\theta \left(1 + \frac{z}{r_1}\right) dz \tag{16}$$

$$Q_\phi = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\phi z} \left(1 + \frac{z}{r_2}\right) dz \tag{17}$$

The form of Eq. (11) and (12) suggests that the stress resultants given by Eq. (13)–(16) can be written in the form

$$N_\phi = N^*_\phi - N_{T\phi} \tag{18}$$

$$N_\theta = N^*_\theta - N_{T\theta} \tag{19}$$

$$M_\phi = M^*_\phi - M_{T\phi} \tag{20}$$

$$M_\theta = M^*_\theta - M_{T\theta} \tag{21}$$

where

$$N^*_\phi = \frac{E}{1-\nu^2}\int_{-\frac{h}{2}}^{\frac{h}{2}} (\epsilon_\phi + \nu\epsilon_\theta)\left(1 + \frac{z}{r_2}\right) dz \tag{22}$$

$$N^*_\theta = \frac{E}{1-\nu^2}\int_{-\frac{h}{2}}^{\frac{h}{2}} (\epsilon_\theta + \nu\epsilon_\phi)\left(1 + \frac{z}{r_1}\right) dz \tag{23}$$

$$M^*_\phi = \frac{E}{1-\nu^2}\int_{-\frac{h}{2}}^{\frac{h}{2}} z(\epsilon_\phi + \nu\epsilon_\theta)\left(1 + \frac{z}{r_2}\right) dz \tag{24}$$

$$M^*_\theta = \frac{E}{1-\nu^2}\int_{-\frac{h}{2}}^{\frac{h}{2}} z(\epsilon_\theta + \nu\epsilon_\phi)\left(1 + \frac{z}{r_1}\right) dz \tag{25}$$

$$N_{T\phi} = \frac{E\alpha}{1-\nu}\int_{-\frac{h}{2}}^{\frac{h}{2}} T(\phi, z)\left(1 + \frac{z}{r_2}\right) dz \tag{26}$$

$$N_{T\theta} = \frac{E\alpha}{1-\nu}\int_{-\frac{h}{2}}^{\frac{h}{2}} T(\phi, z)\left(1 + \frac{z}{r_1}\right) dz \tag{27}$$

$$M_{T\phi} = \frac{E\alpha}{1-\nu}\int_{-\frac{h}{2}}^{\frac{h}{2}} z\, T(\phi, z)\left(1 + \frac{z}{r_2}\right) dz \tag{28}$$

$$M_{T\theta} = \frac{E\alpha}{1-\nu}\int_{-\frac{h}{2}}^{\frac{h}{2}} z\,T(\phi, z)\left(1 + \frac{z}{r_1}\right)dz \qquad (29)$$

Substitution of Eq. (4) and (5) into Eq. (22)–(25) yields, upon neglect of higher order terms,

$$N^*_\phi = D(\bar{\epsilon}_\phi + \nu\bar{\epsilon}_\theta) \qquad (30)$$

$$N^*_\theta = D(\bar{\epsilon}_\theta + \nu\bar{\epsilon}_\phi) \qquad (31)$$

$$M^*_\phi = K(\kappa_\phi + \nu\kappa_\theta) \qquad (32)$$

$$M^*_\theta = K(\kappa_\theta + \nu\kappa_\phi) \qquad (33)$$

where

$$D = \frac{Eh}{1-\nu^2}$$

$$K = \frac{Eh^3}{12(1-\nu^2)}$$

The differential equations of equilibrium as given by Novozhilov (Ref. 2), when expressed in terms of Eq. (18)–(21), become

$$\frac{1}{r_0 r_1}\left[\frac{d}{d\phi}(r_0 N^*_\phi) - N^*_\theta\, r_1 \cos\phi\right] + q_\phi + \frac{Q_\phi}{r_1} = \frac{1}{r_0 r_1}\left[\frac{d}{d\phi}(r_0 N_{T\phi}) - N_{T\theta}\, r_1 \cos\phi\right] \qquad (34)$$

$$\frac{N^*_\phi}{r_1} + \frac{N^*_\theta}{r_2} - \frac{1}{r_1 r_0}\frac{d}{d\phi}(r_0 Q_\phi) = q_z + \frac{N_{T\phi}}{r_1} + \frac{N_{T\theta}}{r_2} \qquad (35)$$

$$\frac{1}{r_0 r_1}\left[\frac{d}{d\phi}(r_0 M^*_\phi) - M^*_\theta\, r_1 \cos\phi\right] - Q_\phi = \frac{1}{r_0 r_1}\left[\frac{d}{d\phi}(r_0 M_{T\phi}) - M_{T\theta}\, r_1 \cos\phi\right] \qquad (36)$$

Similarly, the differential equations of compatibility as given by Novozhilov (Ref. 2), when expressed in terms of the stress resultants, are given by

$$\frac{1}{r_0 r_1}\left\{-\frac{d}{d\phi}[r_0(M^*_\theta - \nu M^*_\phi)] + (M^*_\phi - \nu M^*_\theta)\, r_1 \cos\phi\right\} + \frac{h^2}{12 r_0 r_1^2}\left\{\frac{d}{d\phi}[r_0(N^*_\theta - \nu N^*_\phi)] - (N^*_\phi - \nu N^*_\theta) r_1 \cos\phi\right\} = 0 \qquad (37)$$

$$\frac{M^*_\theta - \nu M^*_\phi}{r_1} + \frac{M^*_\phi - \nu M^*_\theta}{r_2} + \frac{h^2}{12 r_0 r_1}\frac{d}{d\phi}\left\{\frac{1}{r_1}\left[\frac{d}{d\phi}\,(r_0[N^*_\theta - \nu N^*_\phi]) - (N^*_\phi - \nu N^*_\theta)\, r_1 \cos\phi\right]\right\} = 0 \qquad (38)$$

Combining Eq. (36) with Eq. (37) we obtain the following approximate result

$$Q_\phi = \frac{1}{r_1(1+\nu)}\frac{dM^*}{d\phi} - \frac{1}{r_0 r_1}\left[\frac{d}{d\phi}(r_0 M_{T\phi}) - M_{T\theta}\, r_1 \cos\phi\right] \qquad (39)$$

where

$$M^* = M^*_\phi + M^*_\theta$$

Substitution of Eq. (39) into Eq. (34) yields

$$\frac{1}{r_0 r_1}\left[\frac{d}{d\phi}(r_0 N^*_\phi) - N^*_\theta\, r_1 \cos\phi\right] + \frac{1}{r_1^2(1+\nu)}\frac{dM^*}{d\phi} = -q_\phi + \frac{1}{r_0 r_1^2}\left[\frac{d}{d\phi}(r_0 M_{T\phi}) - M_{T\theta}\, r_1 \cos\phi\right] + \frac{1}{r_0 r_1}\left[\frac{d}{d\phi}(r_0 N_{T\phi}) - N_{T\theta}\, r_1 \cos\phi\right] \qquad (40)$$

Similarly, substitution of Eq. (34) and (39) into Eq. (37) yields, upon neglect of higher order terms,

$$\frac{1}{r_0 r_1}\left\{-\frac{d}{d\phi}[r_0(M^*_\theta - \nu M^*_\phi)] + (M^*_\phi - \nu M^*_\theta)\, r_1 \cos\phi\right\} + \frac{h^2}{12 r_1^2}\frac{dN^*}{d\phi} = \frac{h^2(1+\nu)}{12 r_1^2 r_0}\left[\frac{d}{d\phi}(r_0 N_{T\phi}) - N_{T\theta}\, r_1 \cos\phi\right] \qquad (41)$$

where

$$N^* = N^*_\phi + N^*_\theta$$

Substitution of Eq. (39) into Eq. (35) yields

$$\frac{N^*_\phi}{r_1} + \frac{N^*_\theta}{r_2} - q_z - \frac{1}{1+\nu}G(M^*) = \frac{1}{r_0 r_1}\frac{d}{d\phi}\left[-\frac{1}{r_1}\frac{d}{d\phi}(r_0 M_{T\phi}) + M_{T\theta}\cos\phi\right] + \frac{N_{T\phi}}{r_1} + \frac{N_{T\theta}}{r_2} \qquad (42)$$

where

$$G(\cdots) = \frac{1}{r_0 r_1}\frac{d}{d\phi}\left[\frac{r_0}{r_1}\frac{d}{d\phi}(\cdots)\right]$$

Substitution of Eq. (34) and (39) into Eq. (38) yields, upon neglect of certain higher order terms,

$$\frac{M^*_\theta - \nu M^*_\phi}{r_1} + \frac{M^*_\phi - \nu M^*_\theta}{r_2} + \frac{h^2}{12}G(N^*) = \frac{h^2(1+\nu)}{12\, r_1 r_0}\frac{d}{d\phi}\left[\frac{1}{r_1}\frac{d}{d\phi}(r_0 N_{T\phi}) - N_{T\theta}\cos\phi\right] \qquad (43)$$

Making the change of dependent variable in Eq. (40),

$$\Gamma_1 = N^*_\phi - \frac{i}{c}\left(\frac{M^*_\theta - \nu M^*_\phi}{1-\nu^2}\right) \qquad (44)$$

$$\Gamma_2 = N^*_\theta - \frac{i}{c}\left(\frac{M^*_\phi - \nu M^*_\theta}{1-\nu^2}\right) \tag{45}$$

$$\Gamma = \Gamma_1 + \Gamma_2 \tag{46}$$

where

$$c = \frac{h}{\sqrt{12(1-\nu^2)}}$$

$$i = \sqrt{-1}$$

and adding the result to the product of Eq. (41) with $i/c(1-\nu^2)$ leads to

$$\frac{1}{r_0 r_1}\left[\frac{d}{d\phi}(r_0\Gamma_1) - \Gamma_2 r_1 \cos\phi\right] + \frac{ic}{r_1^2}\frac{d\Gamma}{d\phi} = -q_\phi + \frac{1}{r_0 r_1}\left[1 + \frac{ic(1+\nu)}{r_1}\right]\left[\frac{d}{d\phi}(r_0 N_{T\phi}) - N_{T\theta} r_1 \cos\phi\right] + \frac{1}{r_1^2 r_0}\left[\frac{d}{d\phi}(r_0 M_{T\phi}) - M_{T\theta} r_1 \cos\phi\right] \tag{47}$$

Similarly, substituting Eq. (44)–(46) into Eq. (42), and adding the result to the product of Eq. (43) with $-i/c(1-\nu^2)$ yields

$$\frac{\Gamma_1}{r_1} + \frac{\Gamma_2}{r_2} - icG(\Gamma) = q_z + \frac{1}{r_0 r_1}\frac{d}{d\phi} \times\left[-\frac{1}{r_1}\frac{d}{d\phi}(r_0 M_{T\phi}) + M_{T\theta}\cos\phi\right] + \frac{N_{T\phi}}{r_1} + \frac{N_{T\theta}}{r_2} - \frac{ic(1+\nu)}{r_0 r_1}\frac{d}{d\phi}\left[\frac{1}{r_1}\frac{d}{d\phi}(r_0 N_{T\phi}) - N_{T\theta}\cos\phi\right] \tag{48}$$

It follows directly from Eq. (47) and (48) that the thermal stress problem can be considered as an "equivalent surface load problem," the equivalent surface loads being

$$q_{zT} = \frac{1}{r_0 r_1}\frac{d}{d\phi}\left[-\frac{1}{r_1}\frac{d}{d\phi}(r_0 M_{T\phi}) + M_{T\theta}\cos\phi\right] + \frac{N_{T\phi}}{r_1} + \frac{N_{T\theta}}{r_2} - \frac{ic(1+\nu)}{r_0 r_1}\frac{d}{d\phi}\left[\frac{1}{r_1}\frac{d}{d\phi}(r_0 N_{T\phi}) - N_{T\theta}\cos\phi\right] \tag{49}$$

$$q_{\phi T} = -\frac{1}{r_0 r_1}\left[1 + \frac{ic(1+\nu)}{r_1}\right] \times\left[\frac{d}{d\phi}(r_0 N_{T\phi}) - N_{T\theta} r_1 \cos\phi\right] - \frac{1}{r_0 r_1^2}\left[\frac{d}{d\phi}(r_0 M_{T\phi}) - M_{T\theta} r_1\cos\phi\right] \tag{50}$$

Equations (47) and (48) can be combined in order to produce a single equation involving Γ as the dependent variable, and an equation relating Γ_1 to Γ:

$$\frac{d^2\Gamma}{d\phi^2} + \left[\left(2\frac{r_1}{r_2} - 1\right)\cot\phi - \frac{1}{r_1}\frac{dr_1}{d\phi}\right]\frac{d\Gamma}{d\phi} + \frac{ir_1^2}{cr_2}\Gamma = \frac{ir_1^2}{cr_2}F(\phi) \tag{51}$$

$$\Gamma_1 = \frac{ic}{r_1}\cot\phi\frac{d\Gamma}{d\phi} + \frac{1}{r_2\sin^2\phi} \times\left\{C_1 + \int_{\phi''}^{\phi}\left[(q_z + q_{zT})\cos\phi - (q_\phi + q_{\phi T})\sin\phi\right] r_1 r_0\, d\phi\right\} \tag{52}$$

where

$$F(\phi) = r_2(q_z + q_{zT}) - \left(\frac{1}{r_1} - \frac{1}{r_2}\right)\frac{1}{\sin^2\phi} \times\left\{C_1 + \int_{\phi''}^{\phi}\left[(q_z + q_{zT})\cos\phi - (q_\phi + q_{\phi T})\sin\phi\right] r_0 r_1\, d\phi\right\} \tag{53}$$

with C_1 and ϕ'' as arbitrary constants.

For convenience, that portion of the integral appearing in Eq. (52) containing "equivalent surface loads" will be referred to as follows:

$$J(\phi) = \int_{\phi''}^{\phi}(q_{zT}\cos\phi - q_{\phi T}\sin\phi)\, r_1 r_2 \sin\phi\, d\phi \tag{54}$$

Substitution of Eq. (49) and (50) into Eq. (54) yields

$$J(\phi) = -\cos\phi\left[\frac{1}{r_1}\frac{d}{d\phi}(r_0 M_{T\phi}) - M_{T\theta}\cos\phi\right]\Bigg|_{\phi''}^{\phi} + r_0\sin\phi\, N_{T\phi}\Bigg|_{\phi''}^{\phi} - ic(1+\nu)\cos\phi \times\left[\frac{1}{r_1}\frac{d}{d\phi}(r_0 N_{T\phi}) - N_{T\theta}\cos\phi\right]\Bigg|_{\phi''}^{\phi}$$

In particular, if the shell is closed and smooth at the apex (i.e., not pointed), then we can select $\phi'' = 0$, $C_1 = 0$. In this case it can be shown that $r_1(0) = r_2(0)$, and thus it follows that

$$J(\phi) = -\cos^2\phi\left[M_{T\phi} - M_{T\theta}\right] - \frac{r_2}{r_1}\sin\phi\cos\phi\frac{dM_{T\phi}}{d\phi} + r_2\sin^2\phi\, N_{T\phi} - ic(1+\nu)\left[\cos^2\phi\,(N_{T\phi} - N_{T\theta}) + \frac{r_2}{r_1}\sin\phi\cos\phi\frac{dN_{T\phi}}{d\phi}\right] \tag{55}$$

Thus, the problem has been reduced to the integration of Eq. (51).

II. INTEGRATION OF THE SHELL EQUATIONS FOR THE CASE OF A PARABOLIC SHELL OF REVOLUTION SUBJECTED TO AXISYMMETRIC THERMAL LOADING

If the shell is parabolic the principal radii of curvature are given by

$$r_1 = \frac{a}{\cos^3 \phi} \tag{56}$$

$$r_2 = \frac{a}{\cos \phi} \tag{57}$$

where a is twice the focal length of the paraboloid. Substitution of Eq. (56) and (57) into Eq. (51) leads to

$$\frac{d^2\Gamma}{d\phi^2} + 2 \cot 2\phi \frac{d\Gamma}{d\phi} + \frac{ia}{c \cos^5\phi} \Gamma = \frac{iaF(\phi)}{c \cos^5\phi} \tag{58}$$

In the case of thermal loading only, Eq. (53) reduces to

$$F(\phi) = \frac{a}{\cos \phi} q_{zT} + J(\phi) \frac{\cos \phi}{a} \tag{59}$$

The homogeneous part of Eq. (58) is given by

$$\frac{d^2 \Gamma^*}{d\phi^2} + 2 \cot 2\phi \frac{d\Gamma^*}{d\phi} + \frac{ia}{c \cos^5\phi} \Gamma^* = 0 \tag{60}$$

A change of dependent variable

$$\overline{\Gamma} = \Gamma^* \sqrt{\sin 2\phi} \tag{61}$$

in Eq. (60) yields

$$\frac{d^2 \overline{\Gamma}}{d\phi^2} - H^2 \overline{\Gamma} = 0 \tag{62}$$

where

$$H^2 = -\left(1 + \csc^2 2\phi + \frac{ia}{c \cos^5\phi}\right) \tag{63}$$

Novozhilov (Ref. 2) points out that a solution of Eq. (62) which neglects terms of order 0 (c/a) compared with terms of order 0 (1) is given by

$$\frac{\overline{\Gamma}}{\sqrt{\sin 2\phi}} = \Gamma^* = A_1 \exp\left[(i-1)\sqrt{\frac{a}{2c}} \int_0^\phi \frac{d\phi}{\cos^{5/2}\phi}\right] + A_2 \exp\left[(1-i)\sqrt{\frac{a}{2c}} \int_0^\phi \frac{d\phi}{\cos^{5/2}\phi}\right] \tag{64}$$

where A_1, A_2 are arbitrary complex constants of the form

$$A_1 = A_1' + iA_1''$$

$$A_2 = A_2' + iA_2''$$

An approximate particular solution of Eq. (58) is given by

$$\Gamma_{\text{particular}} = F(\phi) \tag{65}$$

Hence it follows that the complete solution of Eq. (58) is

$$\Gamma = \Gamma^* + F(\phi) \tag{66}$$

Various values of the elliptic integral, appearing in Eq. (64),

$$R(\phi) = \int_0^\phi \frac{d\phi}{\cos^{5/2}\phi} \tag{67}$$

have been computed, as shown in Fig. 3. By using Eq. (44), (46), and (67) we obtain

$$N_\phi^* = \sqrt{\frac{c}{2a}} \frac{\cos^{3/2}\phi}{\sin \phi} \times \Big\{ e^{-\xi} \left[(A_1' + A_1'') \sin \xi + (A_1'' - A_1') \cos \xi\right] + e^{\xi} \left[(A_2' + A_2'') \sin \xi + (A_2' - A_2'') \cos \xi\right] \Big\} + \frac{\cos \phi}{a \sin^2\phi} J(\phi) \tag{68}$$

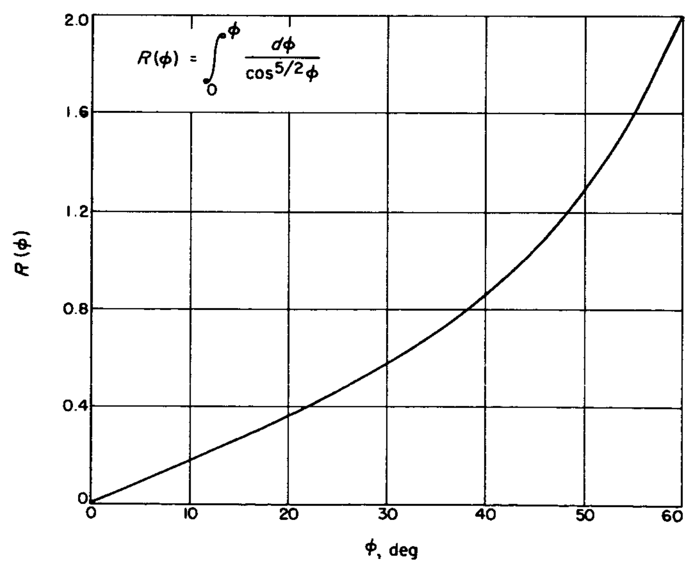

Fig. 3. Plot of $R(\phi)$ vs ϕ

$$N_\theta^* = e^{-\xi}\left[A_1' \cos\xi - A_1'' \sin\xi\right] + e^{\xi}\left[A_2' \cos\xi + A_2'' \sin\xi\right] + F(\phi) - \frac{\cos\phi}{a\sin^2\phi} J(\phi) \tag{69}$$

$$M_\phi^* = -c\left\{e^{-\xi}\left[A_1'' \cos\xi + A_1' \sin\xi\right] + e^{\xi}\left[A_2'' \cos\xi - A_2' \sin\xi\right]\right\} + \frac{h^2}{12a(1+\nu)} \cos^3\phi \cot\phi \frac{dF(\phi)}{d\phi} \tag{70}$$

$$M_\theta^* = -\nu c\left\{e^{-\xi}\left[A_1'' \cos\xi + A_1' \sin\xi\right] + e^{\xi}\left[A_2'' \cos\xi - A_2' \sin\xi\right]\right\} - \frac{h^2}{12a(1+\nu)} \cos^3\phi \cot\phi \frac{dF(\phi)}{d\phi} \tag{71}$$

$$V = \frac{1}{Eh}\sqrt{\frac{a}{2c}}\frac{1}{\sqrt{\cos\phi}} \times \left\{e^{-\xi}(A_1' + A_1'')\cos\xi + (A_1' - A_1'')\sin\xi\right] - e^{\xi}\left[(A_2' + A_2'')\cos\xi + (A_2'' - A_2')\sin\xi\right]\right\} - \frac{\cos^2\phi}{Eh}\frac{dF(\phi)}{d\phi} \tag{72}$$

The latter results are derived on the basis of neglecting terms of order 0 (c/a) compared with the terms of order 0 (1); also, ξ is defined by

$$\xi = \sqrt{\frac{a}{2c}}\, R(\phi)$$

Since the stress resultants are required to remain finite at the apex of the shell, we obtain

$$A_1 = A_1' + iA_1'' = 0$$

or

$$A_1' = 0, A_1'' = 0$$

III. AN ILLUSTRATIVE EXAMPLE

Assume that the edge of the shell is attached to an edge ring of arbitrary cross-sectional shape, at a point on the ring's major horizontal diameter (see Fig. 4), which is in turn attached to supports. If, in particular, the edge ring is sufficiently rigid and the supports are such that the vertical displacement Δ_v given by

$$\Delta_v = w \cos \phi + v \sin \phi \tag{73}$$

is uniform at the edge of the shell, the boundary conditions (see Ref. 3) are given by

$$\Delta_h(\phi_0) = \frac{r_0^2(\phi_0)}{E_1 A} F_h(\phi_0) + r_0 \alpha_1 T\left(\phi_0, \frac{h}{2}\right) \tag{74}$$

$$V(\phi_0) = -\frac{r_0^2(\phi_0)}{E_1 I_1} M_\phi(\phi_0) \tag{75}$$

where

ϕ_0 = the value of ϕ at the edge of the shell

E_1 = Young's modulus for the ring

A = cross-sectional area of the ring

α_1 = coefficient of thermal expansion of the ring

I_1 = moment of inertia of the ring about the centroidal axis in the plane of the ring

and where F_h is the horizontal component of the stress resultants given by

$$F_h = -Q_\phi \sin \phi - N_\phi \cos \phi \tag{76}$$

and the horizontal component of displacement Δ_h is given by

$$\Delta_h = -w \sin \phi + v \cos \phi = r_0 \bar{\epsilon}_\theta \tag{77}$$

Since the shell is subjected only to thermal loading, the vertical component of the stress resultant must vanish. Therefore,

$$Q_\phi = N_\phi \tan \phi \tag{78}$$

thus reducing Eq. (76) to the form

$$F_h = -N_\phi \sec \phi \tag{79}$$

The arbitrary constants A_2', A_2'' can be found by substitution of Eq. (77)–(79), (18)–(20), (68)–(70), and (72) into Eq. (74) and (75), as follows:

$$\begin{aligned} A_2' \delta = {} & e^{-\xi_0} \left\{ \frac{a \tan \phi_0 N_{T\phi}(\phi_0)}{E_1 A \cos \phi_0} + \alpha_1 T\left(\phi_0, \frac{h}{2}\right) + \frac{1}{Eh}\left[\frac{\cot \phi_0}{a \sin \phi_0} J(\phi_0) - F(\phi_0)\right] - \frac{J(\phi_0) \cot \phi_0}{a \sin \phi_0}\left[\frac{a \tan \phi_0}{E_1 A \cos \phi_0} - \frac{\nu}{Eh}\right]\right\} \\ & \times \left\{ \frac{1}{Eh}\sqrt{\frac{a}{2c}} \frac{(\cos \xi_0 + \sin \xi_0)}{\sqrt{\cos \phi_0}} + \frac{ca^2 \tan^2\phi_0 \cos \xi_0}{E_1 I_1} \right\} \\ & - e^{-\xi_0} \left\{ -\frac{\cos^2\phi_0}{Eh} \frac{dF(\phi_0)}{d\phi} - \frac{a^2 \tan^2\phi_0}{E_1 I_1} M_{T\phi}(\phi_0) + \frac{ach^2 \sin \phi_0 \cos^2 \phi_0}{12 E_1 I_1 (1+\nu)} \frac{dF}{d\phi}(\phi_0) \right\} \\ & \times \left\{ \frac{\sin \xi_0}{Eh} + \left[\frac{a \tan \phi_0}{E_1 A \cos \phi_0} - \frac{\nu}{Eh}\right] \sqrt{\frac{c}{2a}} \frac{\cos^{3/2}\phi_0}{\sin \phi_0} (\sin \xi_0 - \cos \xi_0) \right\} \end{aligned} \tag{80}$$

$$\begin{aligned} A_2'' \delta = {} & e^{-\xi_0} \left\{ -\frac{\cos^2\phi_0}{Eh} \frac{dF}{d\phi}(\phi_0) - \frac{a^2 \tan^2\phi_0}{E_1 I_1} M_{T\phi}(\phi_0) + \frac{ach^2 \sin \phi_0 \cos^2\phi_0}{12 E_1 I_1 (1+\nu)} \frac{dF}{d\phi}(\phi_0) \right\} \\ & \times \left\{ \frac{\cos \xi_0}{Eh} + \left[\frac{a \tan \phi_0}{E_1 A \cos \phi_0} - \frac{\nu}{Eh}\right] \sqrt{\frac{c}{2a}} \frac{\cos^{3/2}\phi_0}{\sin \phi_0} (\sin \xi_0 + \cos \xi_0) \right\} \\ & - e^{-\xi_0} \left\{ \frac{a \tan \phi_0 N_{T\phi}(\phi_0)}{E_1 A \cos \phi_0} + \alpha_1 T\left(\phi_0, \frac{h}{2}\right) + \frac{1}{Eh}\left[\frac{\cot \phi_0}{a \sin \phi_0} J(\phi_0) - F(\phi_0)\right] - \frac{J(\phi_0) \cot \phi_0}{a \sin \phi_0}\left[\frac{a \tan \phi_0}{E_1 A \cos \phi_0} - \frac{\nu}{Eh}\right]\right\} \\ & \times \left\{ \frac{1}{Eh}\sqrt{\frac{a}{2c}} \frac{(\cos \xi_0 - \sin \xi_0)}{\sqrt{\cos \phi_0}} - \frac{a^2 c \tan^2 \phi_0 \sin \xi_0}{E_1 I_1} \right\} \end{aligned} \tag{81}$$

where

$$\delta = \frac{1}{E^2 h^2}\sqrt{\frac{a}{2c}} \frac{1}{\sqrt{\cos \phi_0}} + \frac{a^2 c \tan^2 \phi_0}{E E_1 I_1 h} + \left[\frac{a \tan \phi_0}{E_1 A \cos \phi_0} - \frac{\nu}{Eh}\right]\left[\frac{\cot \phi_0}{Eh} + \frac{(ac)^{3/2} \sin \phi_0}{E_1 I_1 \sqrt{2 \cos \phi_0}}\right] \tag{82}$$

and $\xi_0 = \xi(\phi_0)$, which can be found from Fig. 3.

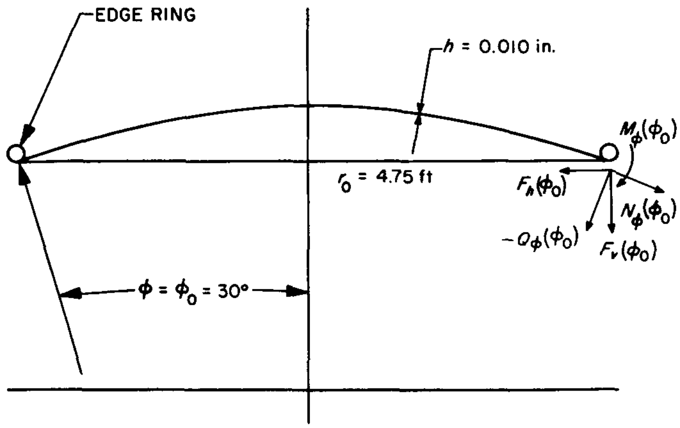

Fig. 4. Configuration of sample problem

If the distribution of temperature change varies linearly through the thickness of the shell such that

$$T = T_1 + zT_2 \tag{83}$$

where T_1, T_2 are independent of ϕ and z, and represent respectively the temperature rise of the midplane and the gradient of the temperature rise, namely,

$$T_1 = \frac{1}{2}\left[T\left(\frac{h}{2}\right) + T\left(-\frac{h}{2}\right)\right] \tag{84}$$

$$T_2 = \frac{1}{h}\left[T\left(\frac{h}{2}\right) - T\left(-\frac{h}{2}\right)\right] \tag{85}$$

Thus we obtain the following relations from Eq. (26)–(29):

$$N_{T\phi} = \frac{Eah}{1-\nu}\left[T_1 + \frac{h^2}{12a}\cos\phi\, T_2\right] \tag{86}$$

$$N_{T\theta} = \frac{Eah}{1-\nu}\left[T_1 + \frac{h^2}{12a}\cos^3\phi\, T_2\right] \tag{87}$$

$$M_{T\phi} = \frac{Eah^3}{12(1-\nu)}\left[\frac{T_1\cos\phi}{a} + T_2\right] \tag{88}$$

$$M_{T\theta} = \frac{Eah^3}{12(1-\nu)}\left[\frac{T_1\cos^3\phi}{a} + T_2\right] \tag{89}$$

In particular, if T_1, T_2 are independent of ϕ, Eq. (55) and (59) take the simple forms

$$J(\phi) = \frac{a\sin^2\phi}{\cos\phi}\frac{Eah}{1-\nu}\left[T_1 + \frac{h^2}{12a}T_2\cos\phi\right] \tag{90}$$

$$F(\phi) = \frac{Eah}{1-\nu}\left[2T_1 + \frac{h^2 T_2}{12a}\cos\phi\,(1+\cos^2\phi)\right] \tag{91}$$

As a numerical example of the use of the latter results, the following problem is considered: a nickel tubular edge ring of 3-in. outer diameter and wall thickness of 0.01 in. is attached at $\phi_0 = 30°$ to a parabolic shell for which $r_0(\phi_0) = 4.75$ ft (thus, the focal length of the shell is 4.12 ft). The shell has the same wall thickness as the edge ring, and is made of the same material. The material properties are given by

$$E = E_1 = 30 \times 10^6 \text{ psi}$$

$$\alpha = \alpha_1 = 7.2 \times 10^{-6}\frac{\text{in.}}{\text{in.} - °\text{F}}$$

$$\nu = 0.37$$

The shell ring system is subjected to a linear distribution of temperature change

$$T = T_1 + zT_2$$

where T_1, T_2 are independent of ϕ.

Substitution of Eq. (88), (90), and (91) into Eq. (80), (81), and (82) yields the following values for constants A_2', A_2'':

$$\frac{A_2'}{T_1} = -e^{-74.57}\left[0.630 \times 10^{-2} + 0.640 \times 10^{-3}k\right] \tag{92}$$

$$\frac{A_2''}{T_1} = e^{-74.57}\left[-16.70 \times 10^{-2} + 13.85 \times 10^{-3}k\right] \tag{93}$$

where

$$k = \frac{T_2}{T_1}$$

A plot of ξ vs ϕ is shown in Fig. 5.

The maximal stresses can be found by substitution of Eq. (92) and (93) into

$$\sigma_\phi = \left(\frac{N_\phi^*}{h} \pm \frac{6}{h^2}M_\phi^*\right) - \left(\frac{N_{T\phi}}{h} \pm \frac{6}{h^2}M_{T\phi}\right) \tag{94}$$

$$\sigma_\theta = \left(\frac{N_\theta^*}{h} \pm \frac{6}{h^2}M_\theta^*\right) - \left(\frac{N_{T\theta}}{h} \pm \frac{6}{h^2}M_{T\theta}\right) \tag{95}$$

Parametric graphs of these quantities and the rotation V/T_1 are shown in Fig. 6–9. The arrows in each of these graphs indicate that the values shown at 28.5° continue uniformly toward the apex of the shell, $\phi = 0°$. The notations σ_ϕ^+, σ_θ^+ refer to the positive sign choice in Eq. (94) and (95) where σ_ϕ^-, σ_θ^- refer to the negative sign choice. It is interesting to note that at a fixed value of k and T_1 the extremal values of the surface stresses and the rotations always occur within the characteristic length of the shell, which is of the order 0(3 in.) or 0 (1°).

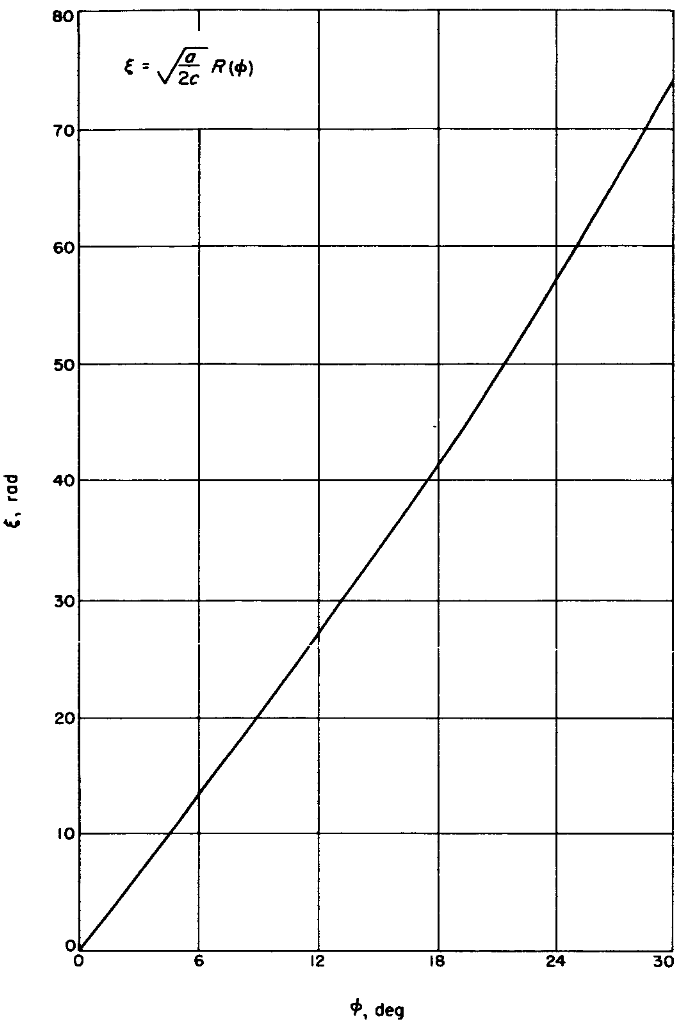

Fig. 5. Plot of ξ vs ϕ for sample problem

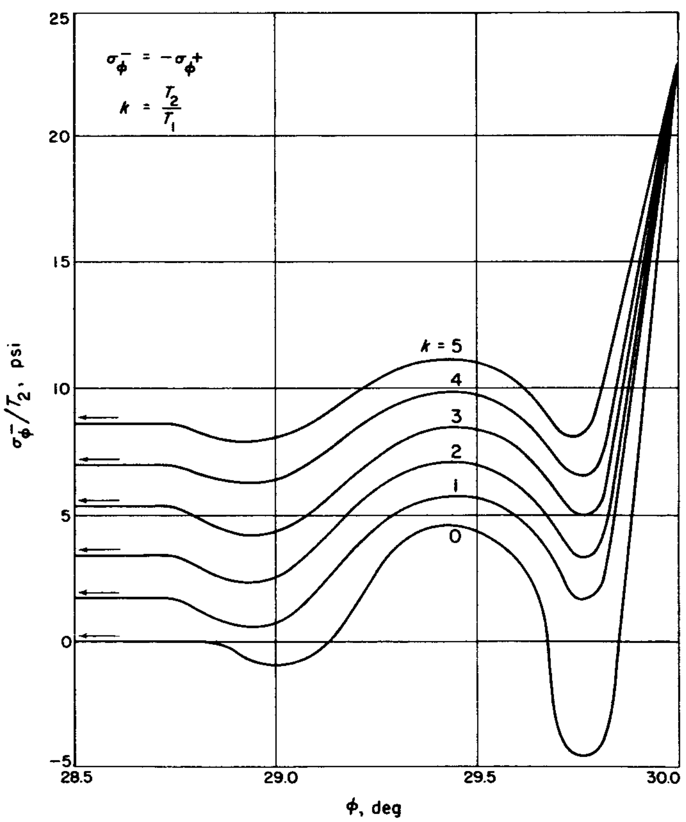

Fig. 6. Plot of σ_ϕ^- vs ϕ for sample problem

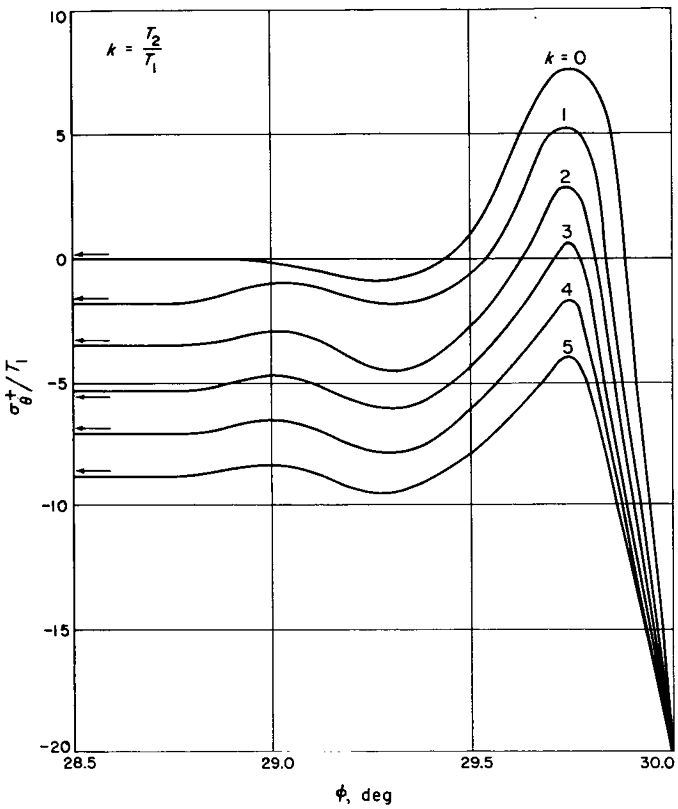

Fig. 7. Plot of σ_θ^+ vs ϕ for sample problem

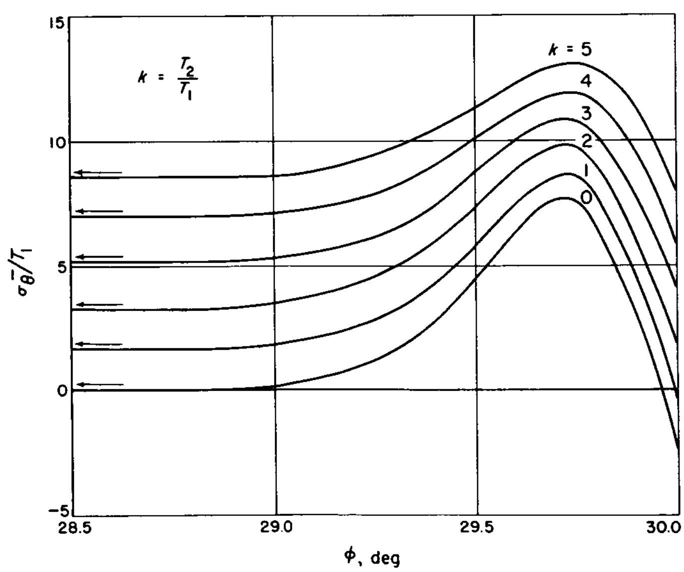

Fig. 8. Plot of σ_θ^- vs ϕ for sample problem

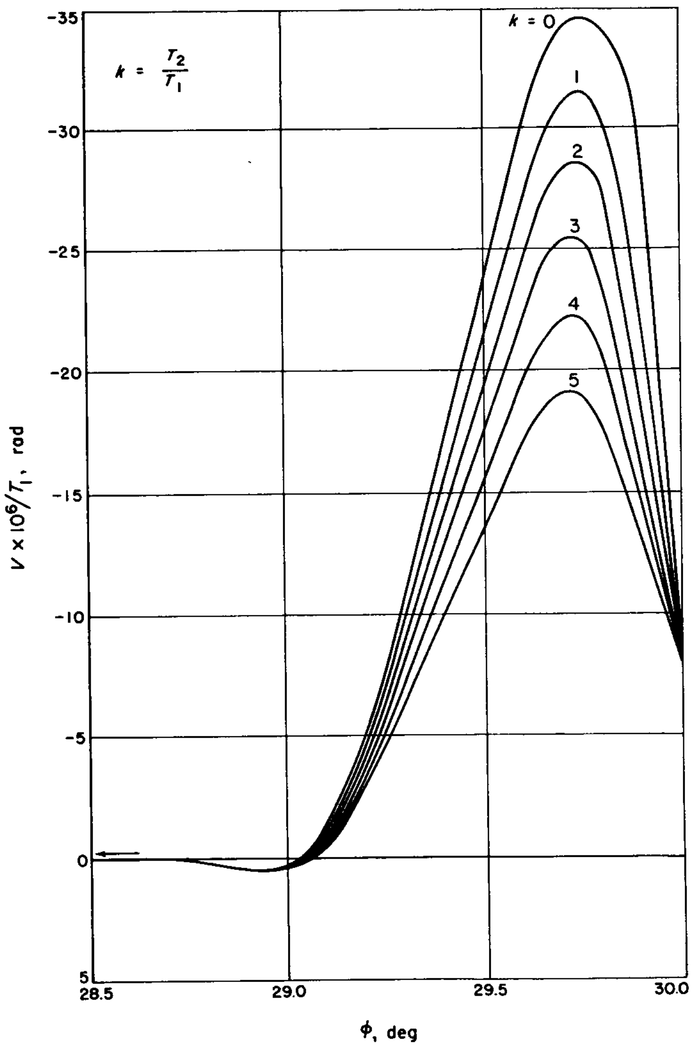

Fig. 9. Plot of rotation vs ϕ for sample problem

IV. CONCLUSION

The axisymmetric, thermoelastic differential equations have been reduced to a single second-order equation involving complex dependent variables. This equation has been approximately solved for a parabolic shell by making use of asymptotic integration. The homogeneous solution is restricted to a region away from the apex $\phi = 0$. The second-order equation can also be solved approximately for other types of shells of revolution by the same method (see Ref. 2).

The stress distribution has been plotted in Fig. 6–8 for the sample problem of a parabolic shell with an edge ring and a temperature gradient through the shell thickness. Figures 6–8 show that the stresses vary rapidly in the region close to the edge $\phi = \phi_0$ and then are uniform throughout the interior of the shell. This is the behavior one would expect. The effect of the parameter k on the stresses and rotation is seen in Fig. 6–9. Increasing values of k represent increasing values of the temperature gradient relative to the midplane temperature. Another interesting result is that the parabolic shell considered behaves much the same as a spherical shell of the same opening angle ϕ_0 and value of r_0 at $\phi = \phi_0$. This is explained by the fact that the radii of curvatures of the parabolic shell considered do not vary much in the region $\phi = 0$ to $\phi_0 = 30°$. The effect of the variation of the radii of curvatures can be seen from Fig. 3. The deviation of the plot $R(\phi)$ vs ϕ from a straight line, which is a measure of the deviation of a parabolic shell from a spherical shell, is seen to be quite small in the region $0 \leqq \phi \leqq 30°$.

ACKNOWLEDGMENT

The author wishes to express his appreciation to R. E. Reed, who read the manuscript and offered many helpful suggestions.

NOMENCLATURE

A_1, A_2	arbitrary complex constants
a	twice the focal length of the parabolic shell
c	$\dfrac{h}{\sqrt{12(1-\nu^2)}}$
D	$\dfrac{Eh}{1-\nu^2}$
E	Young's modulus
F_h	horizontal component of stress resultant
h	thickness of shell
$J(\phi)$	$\int_{\phi''}^{\phi} (q_{zT} \cos\phi - q_{\phi T} \sin\phi) \times r_1 r_2 \sin\phi \, d\phi$
K	$\dfrac{Eh^3}{12(1-\nu^2)}$
k	$\dfrac{T_2}{T_1}$
M_ϕ	meridional bending moment per unit length
M_θ	circumferential bending moment per unit length
M^*	$M_\phi^* + M_\theta^*$
N^*	$N_\phi^* + N_\theta^*$
N_ϕ	meridional force per unit length
N_θ	circumferential force per unit length
$N_{T\phi}, N_{T\theta}, M_{T\phi}, M_{T\theta}$ $N_\phi^*, N_\theta^*, M_\phi^*, M_\theta^*$	defined by Eq. (18) and (29)
Q_ϕ	transverse shear per unit length
q_ϕ	load per unit area tangent to meridian
q_z	radial load per unit area
$q_{zT}, q_{\phi T}$	surface loads equivalent to the thermal load [see Eq. (49) and (50)]
$R(\phi)$	$\int_0^{\phi} \dfrac{d\phi}{\cos^{5/2}\phi}$
r_1	meridional radius of curvature
r_2	circumferential radius of curvature
r_0	$r_2 \sin\phi$
$T(\phi,z)$	difference between the elevated and equilibrium temperatures
T_1, T_2	defined by Eq. (83)–(85)
V	meridional rotation, $\dfrac{1}{r_1}\left(v + \dfrac{dw}{d\phi}\right)$
v	meridional displacement
w	radial displacement
z	thickness variable
α	coefficient of thermal expansion
Γ_1, Γ_2	defined by Eq. (44)–(45)
Γ	$\Gamma_1 + \Gamma_2$
Γ^*	defined by Eq. (60)
$\overline{\Gamma}$	$\Gamma^* \sqrt{\sin 2\phi}$
Δ_h	horizontal component of displacement
Δ_v	vertical component of displacement
ϵ_ϕ	meridional strain
ϵ_θ	circumferential strain
$\epsilon_{\phi z}$	transverse shearing strain
κ_ϕ	change in curvature in meridional direction
κ_θ	change in curvature in circumferential direction
ν	Poisson's ratio
ξ	$\sqrt{\dfrac{a}{2c}}\, R(\phi)$
ξ_0	$\xi(\phi_0)$
σ_ϕ	meridional stress
σ_θ	circumferential stress
ϕ	polar angle between axis of symmetry and normal to shell
ϕ_0	polar angle at edge of shell
ϕ'', C_1	arbitrary constants

REFERENCES

1. Sokolnikoff, I. S., *Mathematical Theory of Elasticity,* McGraw-Hill Book Co., Inc., New York, 1956.
2. Novozhilov, V. V., *The Theory of Thin Shells,* P. Noordhoff, Ltd., Groningen, The Netherlands, 1959.
3. Flügge, W., *Stresses in Shells,* Springer-Verlag, Vienna, 1960.